Free vibrations of steel pole with variable cross-section of arbitrary shape

Static scheme

Cantilever with length - $L = 110\,\mathrm{m}$

Material

Steel

Elastic modulus - $E = 206\,\mathrm{GPa}$ Poisson's ratio - $\nu = 0.3$

Shear modulus - $G = \frac{E}{2\cdot(1+\nu)} = \frac{206\,\mathrm{GPa}}{2\cdot(1+0.3)} = 79.23\,\mathrm{GPa}$

Mass density - $\gamma_s = 7.85\,\frac{\mathrm{t}}{\mathrm{m}^3}$

Cross-section

Section height:

- at bottom - $h_b = 3000\,\mathrm{mm}$
- at top - $h_t = 750\,\mathrm{mm}$
- difference - $\Delta h = h_t - h_b = 750\,\mathrm{mm} - 3000\,\mathrm{mm} = -2250\,\mathrm{mm}$
- as a function of distance to base - $h(x) = h_b + \frac{\Delta h \cdot x}{L}$

Section width:

- at bottom - $w_b = 3000\,\mathrm{mm}$
- at top - $w_t = 750\,\mathrm{mm}$
- difference - $\Delta w = w_t - w_b = 750\,\mathrm{mm} - 3000\,\mathrm{mm} = -2250\,\mathrm{mm}$
- as a function of distance to base - $w(x) = w_b + \frac{\Delta w \cdot x}{L}$

Cross-section shape - **Elliptical tube**

Thickness - $t_b = 20\,\mathrm{mm}$, $t_t = 9\,\mathrm{mm}$, $t(x) = t_b + \frac{(t_t - t_b)\cdot x}{L}$

Function of cross-section outline

$b_i(x;z) = w(x) - 2\cdot t(x)$ $h_i(x;z) = h(x) - 2\cdot t(x)$

$$b_1(x;z) = \frac{2\cdot w(x)}{h(x)} \cdot \sqrt{\left(\frac{h(x)}{2}\right)^2 - \left(z - \frac{h(x)}{2}\right)^2}$$

$$b_2(x;z) = \begin{cases} if\ (z > t(x)) \cdot (z < h(x) - t(x)): & \dfrac{2 \cdot b_i(x;z)}{h_i(x;z)} \cdot \sqrt{\left(\dfrac{h_i(x;z)}{2}\right)^2 - \left(z - \dfrac{h(x)}{2}\right)^2} \\ else: & 0\ \mathrm{m} \end{cases}$$

$$b(x;z) = b_1(x;z) - b_2(x;z)$$

Cross section properties

Area - $A(x) = \int_{0\ \mathrm{mm}}^{h(x)} b_1(x;z)\ dz - \left(\int_{t(x)}^{h(x)-t(x)} b_2(x;z)\ dz\right)$

First moment of area - $S(x) = \int_{0\ \mathrm{mm}}^{h(x)} b_1(x;z) \cdot z\ dz - \left(\int_{t(x)}^{h(x)-t(x)} b_2(x;z) \cdot z\ dz\right)$

Geometrical center- $z_c(x) = \frac{S(x)}{A(x)}$

Second moment of area

$$I_1(x) = \int_{0\ \mathrm{mm}}^{h(x)} b_1(x;z) \cdot (z - z_c(x))^2\ dz \qquad I_2(x) = \int_{t(x)}^{h(x)-t(x)} b_2(x;z) \cdot (z - z_c(x))^2\ dz$$

$$I(x) = I_1(x) - I_2(x)$$

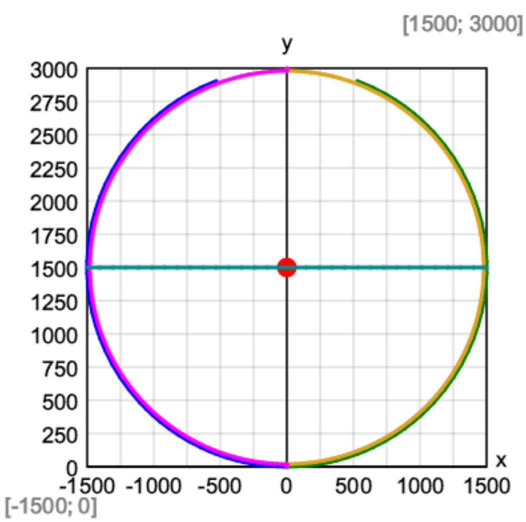
First moment of area under z

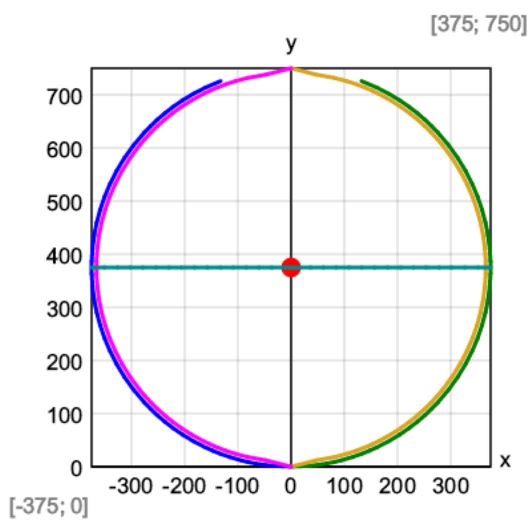
$$S_1(x;z) = \int_{\delta}^{z} b_1(x;\zeta) \cdot (z_c(x) - \zeta)\ d\zeta \qquad S_2(x;z) = \int_{t(x)+\delta}^{z} b_2(x;\zeta) \cdot (z_c(x) - \zeta)\ d\zeta$$

Shear area

$$A_{Q1}(x) = \frac{I_1(x)^2}{\int_{\delta}^{h(x)-\delta} \frac{S_1(x;z)^2}{b_1(x;z)}\ dz} \qquad A_{Q2}(x) = \frac{I_2(x)^2}{\int_{t(x)+\delta}^{h(x)-t(x)-\delta} \frac{S_2(x;z)^2}{b_2(x;z)}\ dz}$$

$$A_Q(x) = A_{Q1}(x) - A_{Q2}(x)$$





Mass

Distributed mass along height - $m(x) = A(x) \cdot \gamma_s$

Solution

Number of nodes - $n_J = 11$

Length of one segment - $\Delta x = \frac{L}{n_J} = \frac{110\,\text{m}}{11} = 10\,\text{m}$

Elevation of node *j* - $x(j) = \Delta x \cdot j$

Bending due to horizontal force $F_j = 1$ at node *j* - $M(j;x) = \mathbf{max}(x(j) - x; 0\,\text{m})$

Flexibility matrix

$$D(i;j) = \left(\int_{0\,\text{m}}^{L} \frac{M(i;x) \cdot M(j;x)}{I(x)}\,dx\right) \cdot \frac{1}{E} + \left(\int_{0\,\text{m}}^{L} \frac{1}{A_Q(x)}\,dx\right) \cdot \frac{1}{G}$$

$$D = \begin{bmatrix} 0.00911 & 0.0219 & 0.0346 & 0.0474 & 0.0601 & 0.0729 & 0.0856 & 0.0984 & 0.11 & 0.12 & 0.14 \\ 0.0219 & 0.0722 & 0.13 & 0.18 & 0.24 & 0.3 & 0.35 & 0.41 & 0.47 & 0.52 & 0.58 \\ 0.0346 & 0.13 & 0.26 & 0.4 & 0.54 & 0.68 & 0.82 & 0.96 & 1.1 & 1.24 & 1.38 \\ 0.0474 & 0.18 & 0.4 & 0.67 & 0.95 & 1.23 & 1.51 & 1.79 & 2.07 & 2.35 & 2.63 \\ 0.0601 & 0.24 & 0.54 & 0.95 & 1.43 & 1.93 & 2.42 & 2.92 & 3.41 & 3.91 & 4.41 \\ 0.0729 & 0.3 & 0.68 & 1.23 & 1.93 & 2.73 & 3.55 & 4.37 & 5.2 & 6.02 & 6.84 \\ 0.0856 & 0.35 & 0.82 & 1.51 & 2.42 & 3.55 & 4.83 & 6.15 & 7.46 & 8.78 & 10.09 \\ 0.0984 & 0.41 & 0.96 & 1.79 & 2.92 & 4.37 & 6.15 & 8.16 & 10.22 & 12.29 & 14.36 \\ 0.11 & 0.47 & 1.1 & 2.07 & 3.41 & 5.2 & 7.46 & 10.22 & 13.38 & 16.63 & 19.87 \\ 0.12 & 0.52 & 1.24 & 2.35 & 3.91 & 6.02 & 8.78 & 12.29 & 16.63 & 21.66 & 26.86 \\ 0.14 & 0.58 & 1.38 & 2.63 & 4.41 & 6.84 & 10.09 & 14.36 & 19.87 & 26.86 & 35.2 \end{bmatrix} \frac{\text{mm}}{\text{kN}}$$

Mass matrix

$$d_{M.j} = \int_{x(j)-\frac{\Delta x}{2}}^{x(j)+\frac{\Delta x}{2}} m(x)\,dx = 1.06\,\text{t}$$

$$M = \begin{bmatrix} 13.01 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 11.43 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9.94 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8.55 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7.26 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6.08 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4.99 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.33 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9 \end{bmatrix}$$

Total mass of structure - $M_{tot} = \mathbf{sum}(\vec{d}_M) = 71.62\,\text{t}$

Eigenvalues

$$M_{sq} = \sqrt{M} = \begin{bmatrix} 3.61 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3.38 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.92 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.47 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2.23 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.77 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.53 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.95 \end{bmatrix}$$

$$C = \mathbf{copy}\left(M_{sq} \cdot D \cdot M_{sq}; \mathbf{symmetric}\left(n_J\right); 1; 1\right) = \mathbf{copy}\left(M_{sq} \cdot D \cdot M_{sq}; \mathbf{symmetric}(11); 1; 1\right)$$

$$= \begin{bmatrix} 0.12 & 0.27 & 0.39 & 0.5 & 0.58 & 0.65 & 0.69 & 0.71 & 0.71 & 0.68 & 0.47 \\ 0.27 & 0.82 & 1.37 & 1.82 & 2.19 & 2.47 & 2.66 & 2.77 & 2.78 & 2.69 & 1.86 \\ 0.39 & 1.37 & 2.59 & 3.7 & 4.6 & 5.29 & 5.79 & 6.07 & 6.14 & 5.98 & 4.14 \\ 0.5 & 1.82 & 3.7 & 5.73 & 7.48 & 8.86 & 9.85 & 10.46 & 10.68 & 10.48 & 7.3 \\ 0.58 & 2.19 & 4.6 & 7.48 & 10.39 & 12.79 & 14.58 & 15.73 & 16.25 & 16.09 & 11.29 \\ 0.65 & 2.47 & 5.29 & 8.86 & 12.79 & 16.57 & 19.54 & 21.57 & 22.62 & 22.66 & 16.03 \\ 0.69 & 2.66 & 5.79 & 9.85 & 14.58 & 19.54 & 24.11 & 27.47 & 29.43 & 29.95 & 21.43 \\ 0.71 & 2.77 & 6.07 & 10.46 & 15.73 & 21.57 & 27.47 & 32.67 & 36.13 & 37.57 & 27.31 \\ 0.71 & 2.78 & 6.14 & 10.68 & 16.25 & 22.62 & 29.43 & 36.13 & 41.73 & 44.85 & 33.37 \\ 0.68 & 2.69 & 5.98 & 10.48 & 16.09 & 22.66 & 29.95 & 37.57 & 44.85 & 50.54 & 39 \\ 0.47 & 1.86 & 4.14 & 7.3 & 11.29 & 16.03 & 21.43 & 27.31 & 33.37 & 39 & 31.82 \end{bmatrix}$$

$$\vec{\lambda} = \mathbf{reverse}\left(\mathbf{last}\left(\mathbf{eigenvals}\left(C \cdot 10^{-3}\right); 3\right)\right) = [0.19 \;\; 0.0172 \;\; 0.00339]$$

Natural frequencies

$$\vec{\omega} = \sqrt{\frac{1}{\vec{\lambda}}} = [2.27 \;\; 7.61 \;\; 17.18]$$

Vibration frequencies

$$\vec{f} = \frac{\vec{\omega}}{2 \cdot \pi} \cdot \text{ Hz} = \frac{\vec{\omega}}{2 \cdot 3.14} \cdot \text{ Hz} = [0.36\,\text{Hz} \;\; 1.21\,\text{Hz} \;\; 2.73\,\text{Hz}]$$

Vibration periods

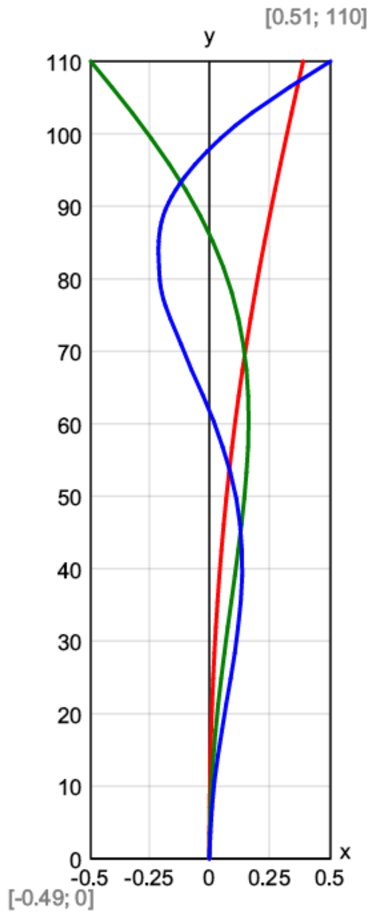
$$\vec{T} = \frac{1}{\vec{f}} = [2.77\,\text{s} \;\; 0.83\,\text{s} \;\; 0.37\,\text{s}]$$

Mode shapes

$$\Phi = \textbf{inverse}(M_{sq}) \cdot \textbf{extract}_{\text{cols}}\left(\textbf{eigenvecs}(C \cdot 10^{-3}); \textbf{range}(n_J; n_J - 2; -1)\right)$$

$$= \textbf{inverse}(M_{sq}) \cdot \textbf{extract}_{\text{cols}}\left(\textbf{eigenvecs}(C \cdot 10^{-3}); \textbf{range}(11; 11 - 2; -1)\right)$$

$$= \begin{bmatrix} 0.00249 & 0.00878 & 0.0192 \\ 0.0103 & 0.0331 & 0.0641 \\ 0.024 & 0.0693 & 0.11 \\ 0.0443 & 0.11 & 0.14 \\ 0.0718 & 0.15 & 0.11 \\ 0.11 & 0.16 & 0.0221 \\ 0.15 & 0.15 & -0.11 \\ 0.2 & 0.0757 & -0.21 \\ 0.26 & -0.0587 & -0.18 \\ 0.32 & -0.26 & 0.0684 \\ 0.39 & -0.49 & 0.51 \end{bmatrix}$$



Comparison with ASCE SEI 7/22 (C26.11-13)

$$\lambda_1 = 1.9 \cdot \left(\exp\left(-4 \cdot \frac{h_t}{h_b}\right)\right) + \frac{6.65}{0.9 + \left(\frac{t_t}{t_b}\right)^{0.67}} = 1.9 \cdot \left(\exp\left(-4 \cdot \frac{750\ \text{mm}}{3000\ \text{mm}}\right)\right) + \frac{6.65}{0.9 + \left(\frac{9\ \text{mm}}{20\ \text{mm}}\right)^{0.67}} = 5.17$$

Fundamental natural frequency (C26.11-12)

$$n_1 = \frac{\lambda_1}{2 \cdot \pi \cdot L^2} \cdot \sqrt{\frac{E \cdot I(0\,\text{m})}{m(0\,\text{m})}} = \frac{5.17}{2 \cdot 3.14 \cdot (110\,\text{m})^2} \cdot \sqrt{\frac{206\,\text{GPa} \cdot I(0\,\text{m})}{m(0\,\text{m})}} = 0.37\,\text{Hz}$$

Fundamental period of vibrations

$$T_1 = \frac{1}{n_1} = \frac{1}{0.37\,\text{Hz}} = 2.72\,\text{s}$$

Comparison with SAP 2000 structural analysis software

Input data:

S T A T I C   L O A D   C A S E S

STATIC CASE	CASE TYPE	SELF WT FACTOR
LOAD1	DEAD	0.0000

J O I N T   D A T A

JOINT	GLOBAL-X	GLOBAL-Y	GLOBAL-Z	RESTRAINTS	ANGLE-A	ANGLE-B	ANGLE-C
1	0.00000	0.00000	0.00000	1 1 1 1 1 1	0.000	0.000	0.000
2	0.00000	0.00000	10.00000	0 0 0 0 0 0	0.000	0.000	0.000
3	0.00000	0.00000	20.00000	0 0 0 0 0 0	0.000	0.000	0.000
4	0.00000	0.00000	30.00000	0 0 0 0 0 0	0.000	0.000	0.000
5	0.00000	0.00000	40.00000	0 0 0 0 0 0	0.000	0.000	0.000
6	0.00000	0.00000	50.00000	0 0 0 0 0 0	0.000	0.000	0.000
7	0.00000	0.00000	60.00000	0 0 0 0 0 0	0.000	0.000	0.000
8	0.00000	0.00000	70.00000	0 0 0 0 0 0	0.000	0.000	0.000
9	0.00000	0.00000	80.00000	0 0 0 0 0 0	0.000	0.000	0.000
10	0.00000	0.00000	90.00000	0 0 0 0 0 0	0.000	0.000	0.000
11	0.00000	0.00000	100.00000	0 0 0 0 0 0	0.000	0.000	0.000
12	0.00000	0.00000	110.00000	0 0 0 0 0 0	0.000	0.000	0.000

F R A M E   E L E M E N T   D A T A

FRAME	JNT-1	JNT-2	SECTION	ANGLE	RELEASES	SEGMENTS	R1	R2	FACTOR	LENGTH
13	1	2	VAR01	0.000	000000	2	0.000	0.000	1.000	10.000
14	2	3	VAR02	0.000	000000	2	0.000	0.000	1.000	10.000
15	3	4	VAR03	0.000	000000	2	0.000	0.000	1.000	10.000
16	4	5	VAR04	0.000	000000	2	0.000	0.000	1.000	10.000
17	5	6	VAR05	0.000	000000	2	0.000	0.000	1.000	10.000
18	6	7	VAR06	0.000	000000	2	0.000	0.000	1.000	10.000
19	7	8	VAR07	0.000	000000	2	0.000	0.000	1.000	10.000
20	8	9	VAR08	0.000	000000	2	0.000	0.000	1.000	10.000
21	9	10	VAR09	0.000	000000	2	0.000	0.000	1.000	10.000
22	10	11	VAR10	0.000	000000	2	0.000	0.000	1.000	10.000
23	11	12	VAR11	0.000	000000	2	0.000	0.000	1.000	10.000

M A T E R I A L   P R O P E R T Y   D A T A

MAT LABEL	MODULUS OF ELASTICITY	POISSON'S RATIO	THERMAL COEFF	WEIGHT PER UNIT VOL	MASS PER UNIT VOL
STEEL	206000000	0.300	1.120E-05	78.500	7.850
CONC	24821128.4	0.200	9.900E-06	23.562	2.401
OTHER	24821128.4	0.200	9.900E-06	23.562	2.401

F R A M E   S E C T I O N   P R O P E R T Y   D A T A

SECTION LABEL	MAT LABEL	SECTION TYPE	DEPTH	FLANGE WIDTH TOP	FLANGE THICK TOP	WEB THICK	FLANGE WIDTH BOTTOM	FLANGE THICK BOTTOM
P00	STEEL		3.000	0.000	0.000	2.000E-02	0.000	0.000
P01	STEEL		2.795	0.000	0.000	1.900E-02	0.000	0.000
P02	STEEL		2.591	0.000	0.000	1.800E-02	0.000	0.000
P03	STEEL		2.386	0.000	0.000	1.700E-02	0.000	0.000
P04	STEEL		2.182	0.000	0.000	1.600E-02	0.000	0.000
P05	STEEL		1.977	0.000	0.000	1.500E-02	0.000	0.000
P06	STEEL		1.773	0.000	0.000	1.400E-02	0.000	0.000
P07	STEEL		1.568	0.000	0.000	1.300E-02	0.000	0.000
P08	STEEL		1.364	0.000	0.000	1.200E-02	0.000	0.000
P09	STEEL		1.159	0.000	0.000	1.100E-02	0.000	0.000
P10	STEEL		0.955	0.000	0.000	1.000E-02	0.000	0.000
P11	STEEL		0.750	0.000	0.000	9.000E-03	0.000	0.000

F R A M E   S E C T I O N   P R O P E R T Y   D A T A

SECTION	AREA	TORSIONAL	MOMENTS OF INERTIA		SHEAR AREAS	
LABEL		INERTIA	I33	I22	A2	A3
P00	0.187	0.416	0.208	0.208	9.462E-02	9.462E-02
P01	0.166	0.319	0.160	0.160	8.375E-02	8.375E-02
P02	0.145	0.241	0.120	0.120	7.356E-02	7.356E-02
P03	0.127	0.178	8.876E-02	8.876E-02	6.398E-02	6.398E-02
P04	0.109	0.128	6.385E-02	6.385E-02	5.508E-02	5.508E-02
P05	9.246E-02	8.898E-02	4.449E-02	4.449E-02	4.679E-02	4.679E-02
P06	7.736E-02	5.985E-02	2.992E-02	2.992E-02	3.917E-02	3.917E-02
P07	6.351E-02	3.839E-02	1.920E-02	1.920E-02	3.217E-02	3.217E-02
P08	5.097E-02	2.329E-02	1.165E-02	1.165E-02	2.584E-02	2.584E-02
P09	3.967E-02	1.307E-02	6.536E-03	6.536E-03	2.014E-02	2.014E-02
P10	2.969E-02	6.629E-03	3.314E-03	3.314E-03	1.509E-02	1.509E-02
P11	2.095E-02	2.876E-03	1.438E-03	1.438E-03	1.068E-02	1.068E-02

F R A M E   S E C T I O N   P R O P E R T Y   D A T A

SECTION	SECTION MODULII		PLASTIC MODULII		RADII OF GYRATION	
LABEL	S33	S22	Z33	Z22	R33	R22
P00	0.139	0.139	0.178	0.178	1.054	1.054
P01	0.114	0.114	0.146	0.146	0.981	0.981
P02	9.295E-02	9.295E-02	0.119	0.119	0.910	0.910
P03	7.440E-02	7.440E-02	9.541E-02	9.541E-02	0.838	0.838
P04	5.853E-02	5.853E-02	7.507E-02	7.507E-02	0.766	0.766
P05	4.501E-02	4.501E-02	5.774E-02	5.774E-02	0.694	0.694
P06	3.375E-02	3.375E-02	4.332E-02	4.332E-02	0.622	0.622
P07	2.449E-02	2.449E-02	3.144E-02	3.144E-02	0.550	0.550
P08	1.708E-02	1.708E-02	2.194E-02	2.194E-02	0.478	0.478
P09	1.128E-02	1.128E-02	1.450E-02	1.450E-02	0.406	0.406
P10	6.941E-03	6.941E-03	8.931E-03	8.931E-03	0.334	0.334
P11	3.835E-03	3.835E-03	4.942E-03	4.942E-03	0.262	0.262

Results:

A S S E M B L E D   J O I N T   M A S S E S   IN GLOBAL COORDINATES

JOINT	UX	UY	UZ	RX	RY	RZ
1	7.067329	7.067329	7.067329	.000000	.000000	.000000
2	13.024974	13.024974	13.024974	.000000	.000000	.000000
3	11.437717	11.437717	11.437717	.000000	.000000	.000000
4	9.949356	9.949356	9.949356	.000000	.000000	.000000
5	8.562760	8.562760	8.562760	.000000	.000000	.000000
6	7.275225	7.275225	7.275225	.000000	.000000	.000000
7	6.089295	6.089295	6.089295	.000000	.000000	.000000
8	5.002586	5.002586	5.002586	.000000	.000000	.000000
9	4.017318	4.017318	4.017318	.000000	.000000	.000000
10	3.131435	3.131435	3.131435	.000000	.000000	.000000
11	2.346830	2.346830	2.346830	.000000	.000000	.000000
12	0.936644	0.936644	0.936644	.000000	.000000	.000000

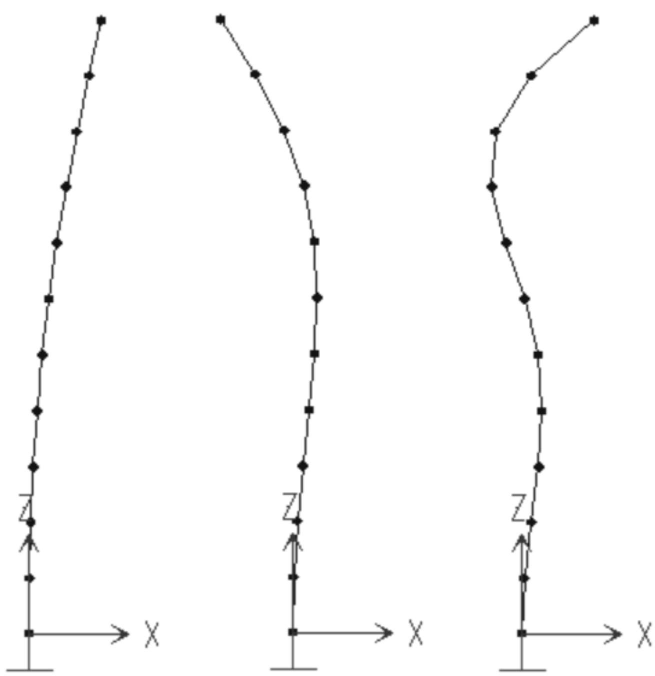
T O T A L   A S S E M B L E D   J O I N T   M A S S E S   IN GLOBAL COORDINATES

	UX	UY	UZ	RX	RY	RZ
TOTAL	78.841467	78.841467	78.841467	.000000	.000000	.000000

M O D A L   P E R I O D S   A N D   F R E Q U E N C I E S

MODE	PERIOD (TIME)	FREQUENCY (CYC/TIME)	FREQUENCY (RAD/TIME)	EIGENVALUE (RAD/TIME)**2
1	2.784527	0.359127	2.256464	5.091629
2	0.831336	1.202883	7.557935	57.122388
3	0.369511	2.706277	17.004037	289.137267

Modal shapes



Φ1   Φ2   Φ3

Differences:

First mode - δ_1 = (2.773 – 2.785)/2.773 = 0.43%

Second mode - δ_2 = (0.826 – 0.831)/0.826 = 0.60%

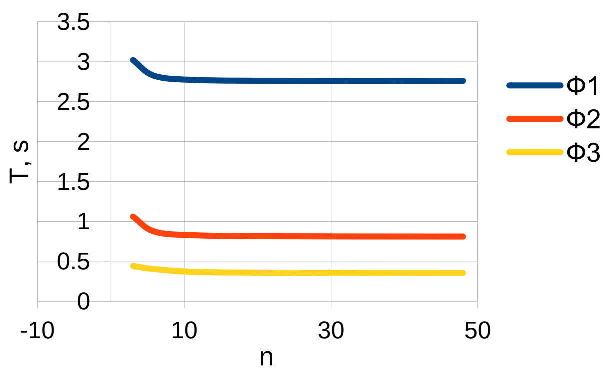
Third mode - δ_3 = (0.367 – 0.370)/0.367 = 0.81%

Convergence of solution in dependance of mesh density:

Number of nodes	Period, s			Difference, %		
	Φ1	**Φ2**	**Φ3**	**Φ1**	**Φ2**	**Φ3**
3	3,02	1,06	0,440	9,42%	30,86%	24,65%
6	2,82	0,870	0,401	2,17%	7,41%	13,60%
12	2,77	0,824	0,365	0,36%	1,73%	3,40%
24	2,76	0,813	0,356	0,00%	0,37%	0,85%
48	2,76	0,810	0,353	0,00%	0,00%	0,00%
96	2,76	0,810	0,353	0,00%	0,00%	0,00%
SAP (11)	2,79	0,831	0,370	0,91%	2,59%	4,82%

Plot of convergence vs number of nodes:

Period



Difference